Concatenation of Rank Metric Codes with Convolutional Codes (for Video Streaming)

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Overview

Rank Metric Codes

Convolutional codes

A concatenated code

Performance
During one *shot* the transmitter injects a number of packets into the network, each of which may be regarded as a row vector over a finite field $\mathbb{F}_{q^m}$. 
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• These packets propagate through the network. Each node creates a random linear combination of the packets it has available and transmits this random combination.
Linear Network Coding

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- These packets propagate through the network. Each node creates a random linear combination of the packets it has available and transmits this random combination.
- Finally, the receiver collects such randomly generated packets and tries to infer the set of packets injected into the network.
Rank metric codes are used in Network Coding

- Rank metric codes are matrix codes $\mathcal{C} \subseteq \mathbb{F}_{q}^{m \times n}$, armed with the rank distance

$$d_{\text{rank}}(X, Y) = \text{rank}(X - Y), \text{ where } X, Y \in \mathbb{F}_{q}^{n \times m}.$$
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- For linear \((n, k)\) rank metric codes over \(\mathbb{F}_{q}^{m}\) with \(m \geq n\) the following analog of the Singleton bound holds,

\[
d_{\text{rank}}(\mathcal{C}) \leq n - k + 1.
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- The code that achieves this bound is called Maximum Rank Distance (MRD). Gabidulin codes are a well-known class of MRD codes.
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- Creating dependencies among the transmitted codewords of different shots can improve the error-correction capabilities (Nobrega, R., Uchoa-Filho (2010), Wachter-Zeh, A., Stinner, M., Sidorenko (2015), Mahmood, R., Badr, A., Khisti(2015)).
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- One standard way to impose correlation of codewords over time is by means of convolution codes.
- We propose to use a **concatenated** code derived by combining a **rank metric** code (as inner code) and a **convolutional code** (as outer code).
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- One standard way to impose correlation of codewords over time is by means of convolution codes.
- We propose to use a concatenated code derived by combining a rank metric code (as inner code) and a convolutional code (as outer code).
- We show how this scheme add complex dependencies to data streams in a quite simple way.
Block codes vs convolutional codes

\[ G \]

\[ ... u_2, u_1, u_0 \overset{G}{\longrightarrow} ... v_2 = u_2 G, \ v_1 = u_1 G, \ v_0 = u_0 G \]

represented in a polynomial fashion

\[ ... + u_2 D^2 + u_1 D + u_0 \overset{G}{\longrightarrow} ... + \underbrace{u_2 G}_{v_2} D^2 + \underbrace{u_1 G}_{v_1} D + \underbrace{u_0 G}_{v_0} \]

substitute \( G \) by \( G(D) = G_0 + G_1 D + \cdots + G_s D^s \)?

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Block codes: \( \mathcal{C} = \{uG\} = \text{Im}_F G \sim \{u(D)G\} = \text{Im}_{F(D)} G \)

Convolutional codes: \( \mathcal{C} = \{u(D)G(D)\} = \text{Im}_{F((D))} G(D) \)
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A matrix $G(D)$ whose rows form a basis for $\mathcal{C}$ is called an encoder. If $\mathcal{C}$ has rank $k$ then we say that $\mathcal{C}$ has rate $k/n$.

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**Remark**

One can also consider the **ring** of polynomials $\mathbb{F}[D]$ instead of Laurent series $\mathbb{F}((D))$ and define $\mathcal{C}$ as a $\mathbb{F}[D]$-module of $\mathbb{F}^n[D]$. 
Historical Remarks

- Convolutional codes were introduced by Elias (1955)
- Became widespread in practice with the Viterbi decoding. Widely implemented codes in (wireless) communications. The field is typically $\mathbb{F}_2$ and the rate and degree are small so that the Viterbi decoding algorithm is efficient.
- Renewed interest for convolutional codes over large alphabets trying to fully exploit the potential of convolutional codes.
- Decoding over the erasure channel is easy (Rosenthal et al. 2012).
The Hamming weight of a polynomial vector

\[ v(D) = \sum_{i \in \mathbb{N}} v_i D^i = v_0 + v_1 D + v_2 D^2 + \cdots + v_\nu D^\nu \in \mathbb{F}[D]^n, \]

defined as \( \text{wt}(v(D)) = \sum_{i=0}^{\nu} \text{wt}(v_i). \)
MDS convolutional codes over $\mathbb{F}$

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The free distance of a convolutional code $C$ is given by,

$$d_{\text{free}}(C) = \min \{ \text{wt}(v(D)) \mid v(D) \in C \text{ and } v(D) \neq 0 \}$$
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- For block codes \( (\delta = 0) \) we know that maximum value is given by the Singleton bound: \( n - k + 1 \)
- This bound can be achieved if \( |\mathbb{F}| > n \)
Theorem

Rosenthal and Smarandache (1999) showed that the free distance of convolutional code of rate $k/n$ and degree $\delta$ must be upper bounded by

$$d_{\text{free}}(C) \leq (n - k) \left( \left\lfloor \frac{\delta}{k} \right\rfloor + 1 \right) + \delta + 1. \quad (1)$$

A code achieving (1) is called **Maximum Distance Separable (MDS)**.
Definition
Another important distance measure for a convolutional code is the $j$th column distance $d^c_j(C)$, (introduced by Costello), given by

$$d^j_H(C) = \min \left\{ \text{wt}(v_{[0,j]}(D)) \mid v(D) \in C \text{ and } v_0 \neq 0 \right\}$$

where $v_{[0,j]}(D) = v_0 + v_1 D + \ldots + v_j D^j$ represents the $j$-th truncation of the codeword $v(D) \in C$. 
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The column distances satisfy

\[
d^0_H \leq d^1_H \leq \cdots \leq \lim_{j \to \infty} d^j_H(C) = d_{\text{free}}(C) \leq (n - k) \left( \left\lfloor \frac{\delta}{k} \right\rfloor + 1 \right) + \delta + 1.
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If $C$ has the best possible column distance profile then it is called Maximum Distance Profile (MDP).

The construction of MDP boils down to the construction of Superregular matrices (difficult over small fields).
Performance over the erasure channel

Theorem (Rosenthal et al. 2012)

Let $C$ be convolutional code and $d_{H}^{j_{0}}$ its $j = j_{0}$ -th column distance.

If in any sliding window of length $(j_{0} + 1)n$ at most $d_{H}^{j_{0}} - 1$ erasures occur then we can recover completely the transmitted sequence.
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**Remark**

The best scenario happens when the convolutional code is MDP.
A [202, 101] MDS block code can correct 101 erasures in a window of 202 symbols (recovering rate \( \frac{101}{202} \)): \( \Rightarrow \) cannot correct this window.

A (2, 1, 50) MDP convolutional code has also 50% error capability. \( (L + 1)n = 101 \times 2 = 202 \). Take a window of 120 symbols, correct and continue until you correct the whole window.

We have flexibility in choosing the size and position of the sliding window.
The proposed concatenation scheme

- \( \mathcal{C}_O \) an \((K, k, \delta)\) convolutional code over the field \( \mathbb{F}_{q^{mn}} \)
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u(D) = u_0 + u_1 D + u_2 D^2 + \cdots \in \mathbb{F}_{q^{mn}}[D]^k \text{ the information vector.}
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\cdots + u_2 D^2 + u_1 D + u_0 \xrightarrow{G_O(D)} \cdots + v_2 D^2 + v_1 D + v_0
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\]

We divide

$v_i = (v_i^0, v_i^1, \ldots, v_i^{K-1})$
We identify $v_i^j \in \mathbb{F}_{q^m}$ with a vector $V_{i}^j \in \mathbb{F}_{q^n}$ (for a given basis of $\mathbb{F}_{q^m}$) and write

$$V_i = (V_i^0, V_i^1, \ldots, V_i^{K-1})$$

and therefore

$$V(D) = V_0 + V_1 D + V_2 D^2 + \cdots \in \mathbb{F}_{q^m}[D]^K.$$
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$$V(D) = V_0 + V_1 D + V_2 D^2 + \cdots \in \mathbb{F}_{q^m}[D]^K.$$  

Finally, the **codewords** $X(D)$ of $C$ are obtained through the matrix $G_I \in \mathbb{F}_{q^m}^{n \times N}$ in the following way:

$$X_i^j = V_i^j G_I,$$

$$X_i = (X_i^0, X_i^1, \ldots, X_i^{K-1})$$

and

$$X(D) = X_0 + X_1 D + X_2 D^2 + \cdots \in C \subset \mathbb{F}_{q^m}^{m \times n}[D]^N.$$
Distance notions

Definition

The \textit{sum rank distance} of $C$ is defined as

$$d_{SR}(C) = \min_{0 \neq X(D) \in C} \text{rank} \ (X(D)) := \min_{0 \neq X(D) \in C} \sum_{i \geq 0} \text{rank} \ (X_i)$$

where

$$\text{rank} \ (X_i) := \sum_{j=0}^{K-1} \text{rank} \ (X_i^j).$$

And the \textit{column sum rank distance} of $C$ is defined as

$$d_{SR}^j(C) = \min_{X(D) \in C \text{ and } X_0^0 \neq 0} \sum_{i=0}^{j} \text{rank} \ (X_i),$$
We assume throughout the paper that $m > N$.

**Theorem**

*The Sum Rank distance of the concatenated code $\mathcal{C}$ is*

$$d_{SR}(\mathcal{C}) = d_{\text{free}}(\mathcal{C}_O) \times d_{\text{rank}}(\mathcal{C}_I).$$

**Theorem**

*The Column Sum Rank distance of $\mathcal{C}$ is*

$$d_{SR}^j(\mathcal{C}) = d_{H}^j(\mathcal{C}_O) \times d_{\text{rank}}(\mathcal{C}_I).$$

where $d_{H}^j(\mathcal{C}_O)$ is the column distance of $\mathcal{C}$. 
Performance

- The rank metric code (inner code) takes care of the errors and erasures (deletion and injection of packets) during the transmission.

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Theorem

If in any sliding window of $\mathcal{C}$ of length $(j_0 + 1)$ at most $d_{SR}^{j_0} - 1$ packet losses occur, then we can completely recover the information sequence.
Previous Theorem has some drawbacks:

- Only **necessary conditions**. There are erasure patterns that do not satisfy the condition of the Theorem but still can be recovered.

- One has to **wait** for the whole sequence to arrive. One would rather decode **sequentially**.
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**Theorem**

Let $d^0_H(C_o), d^1_H(C_o), \ldots, d^L_H(C_o)$ be the distance profile of $C_o$. Let $L_i$ be the number of packet losses at time instant $i$. Assume that we have been able to correctly decode up to an instant $t - 1$. Then, we can completely decode up to an instant $T$, $t \leq T$ iff

$$ \sum_{i=0}^{s} L_{T-i+t} \leq d^s_H(C_o) d_{\text{Rank}}(C_I) - 1 \text{ for } s = 0, 1, \ldots, T. $$
Remains to be investigated

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- Simulations: Rate of success in recovering for a given probability of losing a packet.
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- The last Theorem suggests an algorithm: Exploit the structure of the equations to solve.
- **Simulations**: Rate of success in recovering for a given probability of losing a packet.
- Little is known about how to construct good convolutional codes:
  - MDP or equivalently: Construction of superregular matrices over small fields.
  - Constructions tailor-made to deal with *burst of erasures* (in both Hamming and Rank metric)
Thanks for the (financial) support!